Biot-Savart correlations in layered superconductors.

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We discuss the superconductor to normal phase transition in an infinite–layered, type–II superconductor in the limit where the Josephson coupling between layers is negligible. We model each layer as a neutral gas of thermally–excited pancake vortices and assume the dominant interlayer coupling is the electromagnetic inter-action between the screening currents induced by these vortices. Our main result, obtained by exactly solving the leading order renormalization group flow, is that the phase transition in this model is a Kosterlitz–Thouless transition despite being a three–dimensional system. While the transition itself is driven by the unbinding of two–dimensional pancake vortices, an RG analysis of the low temperature phase and a mean–field theory of the high temperature phase reveal that both phases possess three–dimensional correlations. An experimental consequence is that the jump in the measured in–plane superfluid stiffness, a universal quantity in 2d Kosterlitz–Thouless theory, receives a small non–universal correction (of order 1% in $Bi_2Sr_2CaCu_2O_{8+x}$). This overall picture places some claims expressed in the literature on a more secure analytical footing and resolves some conflicting views.

I. INTRODUCTION

In this paper, we revisit the problem of the superconductor–normal phase transition of a layered, type–II superconductor in the limit where Josephson coupling between layers is negligible. We model this system as an infinite stack of superconducting planes, each layer containing a neutral gas of thermally–excited, two–dimensional pancake vortices.[1] We take the viewpoint that the dominant mechanism coupling the layers is the long–range electromagnetic interaction[2] between the screening currents induced by these vortices. We expect this model to be relevant to layered superconductors where the dominant mechanism by which the superconductivity is lost (as the temperature is raised) is the loss of long–range order in the phase of the order parameter.[3] Candidate materials include the underdoped high–T_c cuprates as well as layered structures made from conventional type–II superconductors.

One motivation for considering this model is that investigations of the superconductor–normal phase transition in the cuprates have revealed 3dXY critical exponents, a hallmark of Josephson coupling between the planes, in only one case: optimally–doped $YBa_2Cu_3O_{6+x}$, the least anisotropic of these materials.[4] In underdoped $Bi_2Sr_2CaCu_2O_{8+x}$, the most anisotropic of these compounds, signatures suggestive of a Kosterlitz–Thouless (KT) transition have been seen[5,6,7] while recent measurements[8] on underdoped $YBa_2Cu_3O_{6+x}$, which is substantially more anisotropic than the optimally–doped material though less so than underdoped $Bi_2Sr_2CaCu_2O_{8+x}$, show a transition that is neither KT nor 3dXY nor any obvious interpolation in between.[37] While these observations do not conclusively show that Josephson coupling can be neglected[9,10], they do suggest that investigations of different mechanisms for coupling the layers might be a fruitful line of attack if there is reason to suspect that Josephson coupling is very small. The Biot–Savart interaction between screening currents in different layers is a long–range, three–dimensional coupling which is always present though its influence is usu-ally assumed to be small compared to the Josephson term.

A second motivation is a recent experiment[11] on $La_{2-x}Ba_xCuO_4$ near its stripe-ordered state at $x = 1/8$ where a 2d superconducting phase has been observed below an apparent Kosterlitz–Thouless transition temperature. One proposal[12] suggests that under the right circumstances, the superconducting state can occur with a finite wave vector, where the periodicity is in the same direction as the charge order but with double the period. Since the stripes in adjacent layers orient with a relative angle of $90°$, an orthogonality argument implies that the Josephson coupling between first, second, and third neighbor planes is cancelled exactly and further estimates[12] imply the residual terms are extremely small. In such a scenario, we expect our model to be directly applicable to the experiments.[38] In a similar vein, there are a series of somewhat older experiments[13,14] on various cuprates in a dc flux transformer geometry where the observation of effectively 2d vortices was explained as the cutting of 3d vortex lines. Since Josephson coupling is the strongest reason why pancake vortices tend to form 3d stacks/lines (as the penalty for breaking a line of pancakes bound by the Josephson inter-action would be proportional to the system size L) one way to interpret the experiment is that the Josephson coupling is effectively "cancelled" by the applied field. Once again, in such a case we would expect our model to apply.

Our main result is that the superconductor to normal phase transition in a layered superconductor with an infinite number of electromagnetically–coupled layers is still a Kosterlitz–Thouless transition. The mechanism resembles the single–layer problem in that (a) the transition occurs through the unbinding of two–dimensional pancake vortices and (b) the screening within an individual layer at temperatures above T_{KT} is not significantly different from an isolated two–dimensional system. However, both the low and high temperature phases have three–dimensional correlations and the jump in the in–plane superfluid stiffness, as inferred from a penetration depth measurement, receives a small non–

universal correction (of order 1%). Our results are obtained via a renormalization group study of the low temperature phase and a Debye–Huckel mean field theory of the high temperature state.

The electromagnetic coupling may be formulated as an interaction between the pancake vortices.[1,2,15,16,17,18,19] The basic mechanism is that a vortex in one of the layers induces screening currents in the same and in other layers, which cause Biot–Savart forces on the other vortices. For a single, literally two–dimensional layer, this vortex–vortex interaction is screened at distances larger than the magnetic penetration depth λ. For distances much less than λ, but greater than the coherence length ξ, the interaction energy of two vortices of the same sign will be repulsive and scale logarithmically with separation. However, if the layer has a small but nonzero thickness d, the effective screening length becomes $\Lambda' = 2\lambda^2/d \gg \lambda$.[15,19] Λ' often exceeds the sample sizes considered in experiment and in such cases, the interaction is effectively logarithmic.

For a layered system, the relevant screening length becomes $\Lambda = 2\lambda_\parallel^2/s$, where $\lambda_\parallel$ is the in–plane magnetic penetration depth and s the layer spacing. However, for an infinite number of layers, the interaction between two vortices of the same sign in the same layer is logarithmic at *all* length scales, not just for separations smaller than Λ. The difference stems from the fact that in the infinite layer problem, currents in the other layers guide a vortex's magnetic flux radially out to infinity within a disk of thickness $\lambda_\parallel$ while in the single layer problem, the flux spreads over all space.[1] For two vortices of the same sign in different layers, it turns out the interaction is also logarithmic at large distances but *attractive*.[1,17] Therefore, the interlayer coupling favors pancake vortices of the same sign aligning into stacks. This is qualitatively what happens with Josephson coupling except the attractive force keeping the pancakes aligned is now logarithmic and long–ranged, instead of linear and short–ranged.

Because the interlayer interaction is logarithmic, it was conjectured[20] that the phase transition should be in the Kosterlitz–Thouless universality class despite being a three–dimensional system. A number of important renormalization group studies[21,22,23,24], emphasizing the role of pancake vortices, explored the issue in greater detail. In each case, the phase transition was investigated through numerical studies of the resulting flow equations, where the interlayer interactions were treated at varying levels of approximation and detail (in each case, the Kosterlitz–Thouless equations occurred as leading terms when the interlayer interaction was treated perturbatively). Refs. [21,22,23] provided support for the KT scenario while Ref. [24], which was prima facie the most complete study, reached a very different conclusion: runaway RG flows in a very narrow temperature range close to T_{KT} appeared to signal a three–dimensional critical region[24] or perhaps a first–order transition. This motivated the present work. Our central result, obtained by exactly solving the leading order RG flows which occur when the interlayer interaction is treated non–perturbatively, is that the phase transition is, indeed, in the Kosterlitz–Thouless universality class.

There are a number of reasons, in addition to three–dimensionality, why a KT transition is not a foregone conclusion for this model. In Ref. [22], it was noted that while the interlayer logarithms individually come with much smaller coefficients than the in–plane logarithm, the infinite set of couplings obey a "sum rule" (see Eq. (2.5)) which follows from flux conservation. Therefore, there is the possibility that the collective effect of a large number of layers could influence the critical properties. In a similar vein, in Ref. [25] it was noted that configurations involving stacks of vortices where one or both ends terminate *inside* the superconductor, which are topologically forbidden in the presence of Josephson coupling (since vortex lines may then only terminate on the surface of the material), should be accounted for in its absence.

In the next Section, we review some basic facts about the Biot–Savart interaction. In Section III we present an analytical theory of the phase transition using an extension of the 2d momentum shell renormalization group[26]. Accounting for configurations involving stacks of vortices, as discussed above, we re–derive the coupled set of flow equations obtained in Ref. [24] and establish this set as an accurate description of the low temperature physics. We explicitly solve this set and find that the phase transition is, in fact, in the Kosterlitz–Thouless universality class. In Sections IV and V, we discuss the low and high temperature phases, the latter using a Debye–Huckel mean field theory. We conclude with a summary. Technical aspects of the calculation are discussed in three appendices.

II. THE BIOT-SAVART INTERACTION

In this Section, we give a physical discussion of the Biot–Savart interaction and introduce our model Hamiltonian. To make our assumptions clear, in Appendix A we review how this interaction formally arises from a Ginzburg–Landau type free energy functional.

We model our system as an infinite stack of superconducting planes, where the stacking is in the z-direction and the positions of the planes are given by $z_n = ns$ where n is an integer and s the interlayer spacing. A two-dimensional "pancake" vortex[27] of strength m_1 (m_1 is an integer) placed at the origin of layer $n = 0$ will induce an azimuthal screening current $m_1 K_\phi(\rho, ns)$ in layer n where ρ is the cylindrical radial coordinate. A vortex of strength m_2 located at position (ρ, ns) will feel a radial Lorentz force given by $F_\rho(\rho, ns) = K_\phi(\rho, ns)m_1 m_2 \phi_0/c$ where $\phi_0 = \frac{hc}{2e}$.

These screening currents were computed in Ref. 1 in the limit where s is small compared to the in-plane penetration length $\lambda_\parallel$ and for distances ρ large compared to s and the in-plane coherence length $\xi_\parallel$:

$$K_\phi(\rho, ns = 0) = \frac{c\phi_0 s}{8\pi^2\lambda_\parallel^2\rho}[1 - \frac{s}{2\lambda_\parallel}(1 - e^{-\rho/\lambda_\parallel})] \quad (2.1)$$

$$K_\phi(\rho, ns \neq 0) = -\frac{c\phi_0 s^2}{16\pi^2\lambda_\parallel^3\rho}(e^{-\frac{|ns|}{\lambda_\parallel}} - e^{-\frac{\sqrt{\rho^2+(ns)^2}}{\lambda_\parallel}}) \quad (2.2)$$

Eq. (2.1) indicates that to leading order, the in-plane screening current $K_\phi(\rho, 0) \sim \frac{1}{\rho}$. This implies an in-plane vortex-

vortex interaction potential where vortices of the same sign repel each other logarithmically with distance, as in the 2dXY model.[28] If, in addition to $\rho \gg s, \xi_\parallel$, we also assume that $\rho \gg \lambda_\parallel, |ns|$, then Eq. (2.2) indicates that the *out-of-plane* screening current $K_\phi(\rho, ns) \sim -\frac{1}{\rho}$. This implies that the interaction between two vortices in different planes also varies logarithmically with distance but vortices of the same sign now *attract*. Therefore, in this limit, the interaction between two vortices of strengths m_1 and m_2 at positions $(\mathbf{x_1}, n_1 s)$ and $(\mathbf{x_2}, n_2 s)$ is given by:

$$V_{12} \approx -q^2 m_1 m_2 \alpha_{|n_1 - n_2|} \ln\left(\frac{|\mathbf{x_1} - \mathbf{x_2}|}{\tau}\right) \qquad (2.3)$$

where

$$\alpha_n \approx \begin{cases} -\sum_{n \neq 0} \alpha_n \approx 1 - \frac{s}{2\lambda_\parallel} & \text{if } n = 0 \\ -\frac{s}{2\lambda_\parallel} e^{-\frac{s}{\lambda_\parallel}|n|} & \text{if } n \neq 0 \end{cases} \qquad (2.4)$$

, $q = \sqrt{\frac{\phi_0^2}{8\pi^2 s}(\frac{s}{\lambda_\parallel})^2}$, and τ is a nominal short distance length scale (of order $\lambda_\parallel$).

Eq. (2.3) is a model of the Biot-Savart interaction which was considered in Refs. 22 and 24, and is the model considered in this paper. A noteworthy feature of this interaction is that the coupling constants obey a "sum rule":

$$\sum_n \alpha_n = 0 \qquad (2.5)$$

This is not an accident but follows from flux quantization[22] and is related to an important feature of the full interaction (Eqs. (2.1), (2.2), and (A13)): the current distribution of an infinite stack of pancake vortices, one in each layer, is exponentially screened at distances $\rho \gg \lambda_\parallel$,[1] in analogy with the well-known result for a vortex line in a bulk three–dimensional superconductor.[29] In contrast, for a stack of uncoupled two–dimensional layers, the current in each layer would decay according to the Pearl criterion[15]: as $\sim \frac{1}{\rho}$ for distances $\rho < \Lambda$ and as $\sim \frac{1}{\rho^2}$ for $\rho \gg \Lambda$, where $\Lambda = \frac{2\lambda_\parallel^2}{d}$, d being the layer thickness.

The electromagnetic interaction causes (same charge) pancake vortices in different layers to preferentially align into stacks, which is phenomenologically what happens with Josephson coupling except that now the aligning force is logarithmic and long-ranged, as opposed to linear and between neighboring layers in the Josephson coupled case. In particular, even though $\alpha_1 \ll \alpha_0$, which naively suggests that only the in-plane interaction is important, the α's decay very slowly and the sum rule indicates that the combined effect of *many* layers could possibly lead to three-dimensional effects. We will return to this issue in the next Section.

We conclude this Section by discussing some of the approximations inherent in Eq. (2.3). In the high temperature superconductors, typical orders of magnitude[1] are $s \approx 12$ Å and $\lambda_\parallel \approx 1400$ Å so the smallness of $\frac{s}{\lambda_\parallel} \approx 10^{-2}$ assumed in the derivation is met in practice. In and near the low temperature

superconducting phase, we expect the density of (thermally–excited) pancake vortices in each layer to be small and hence the characteristic vortex-vortex separation $\bar{\rho}$ to be large. In our analysis, we assume that $\bar{\rho}$ is large compared to other characteristic lengths, such as $\lambda_\parallel$, which is one of the conditions for the interlayer logarithmic form to be valid. Also, in an infinite layer system, the logarithmic approximation will break down for interlayer separations $|ns| \approx \bar{\rho}$. However, the length scale associated with the convergence of the sum rule (Eq. (2.5)) is of order $\lambda_\parallel$ so, if $\bar{\rho} \gg \lambda_\parallel$, we expect the net contribution of these farthest layers to be small regardless of whether the full (Eq. (A13)) or simplified (Eq. (2.3)) interaction is used. Therefore, in this paper, we will approximate the Biot-Savart interaction by its long distance form, which permits us to use Coulomb gas techniques to analyze the partition function. However, we will retain some aspects of the short distance physics, i. e. on length scales shorter than $\lambda_\parallel$, by introducing fugacity variables as discussed in the next Section.

III. RENORMALIZATION GROUP ANALYSIS

The partition function of a layered superconductor, where each layer contains a neutral gas of thermally–excited pancake vortices is:

$$\mathcal{Z} = \sum_{\{N_{k,l}\}} \prod_{k,l} y_{k,l}^{N_{k,l}} \sum_{\{c\}} \exp(-\beta \sum_{i \neq j} \tilde{V}_{ij}), \qquad (3.1)$$

where $y_{k,l} = \exp(-\beta E_{k,l})$, β being the inverse temperature and $E_{k,l}$ the energy cost of creating a pancake vortex of type k in layer l, the species label k denoting both strength and sign. We assume the layers are equivalent which implies the fugacities are the same in each layer, i.e. $y_{k,l} = y_k$. $N_{k,l}$ is the number of type k vortices in layer l. The sum on $\{N_{k,l}\}$ is over layer occupations which satisfy charge (vortex) neutrality in each layer. The sum on $\{c\}$ is over spatial configurations of vortices consistent with the set $\{N_{k,l}\}$.[39] $\tilde{V}_{ij}$ is the vortex–vortex interaction which includes a hard–core constraint that two vortices in the *same* layer must be separated by a distance τ of order $\xi_\parallel$, the in–plane coherence length.

At this stage, $\tilde{V}_{ij}$ is the exact vortex–vortex interaction. To make analytical progress, we separate $\sum_{i \neq j} \tilde{V}_{ij}$ into two parts, $\sum_{i \neq j}^{<} \tilde{V}_{ij} + \sum_{i \neq j}^{>} \tilde{V}_{ij}$, corresponding to the contribution from pairs of vortices with in–plane separation $\rho < \lambda_\parallel$ and $\rho > \lambda_\parallel$ respectively. The latter interaction is given by Eq. (2.3) and in a dilute system will apply for most of the pairs (with the caveats mentioned the previous Section). We approximate the shorter–distance physics in two steps. The possibility of having two vortices in the same plane separated by a distance less than $\lambda_\parallel$ can be accounted for by suitably redefining the fugacity variables.[40] To approximate the interaction between two closely spaced vortices in different layers, we introduce a new set of fugacity variables $\{w_{ab;ij}\}$, where $w_{ab;ij} = \exp(-\beta E_{ab;ij})$, $E_{ab;ij}$ being the interaction energy of having a pancake vortex of type b in layer j directly above a pancake vortex of type a in layer i. For equivalent layers,

$w_{ab;ij}$ will depend only on $|i-j|$ (and the strengths a and b). With these approximations, Eq. (3.1) becomes:

$$\mathcal{Z} = \sum_{\{N_{k,l};N_{ab;ij}\}} \prod_{k,l} y_{k,l}^{N_{k,l}} \prod_{ab;ij} w_{ab;ij}^{N_{ab;ij}} \sum_{\{c\}} \exp(-\beta \sum_{i\neq j}^{>} V_{ij}),$$
(3.2)

where $N_{ab;ij}$ is the number of pairs of vortices where the first member is a vortex of type a in layer i which is directly below the second member, a vortex of type b in layer j. The sum over $\{c\}$ is now over those spatial configurations of pancakes consistent with a set of (charge–neutral) layer occupations $\{N_{k,l}\}$ and interlayer patterns $\{N_{ab;ij}\}$. The sum inside the exponential is over pairs of vortices that are laterally separated by at least $\lambda_\parallel$ and their interaction V_{ij} is given by Eqs. (2.3)–(2.4).

There is another way of viewing Eq. (3.2), which is more in-line with previous treatments of a vector Coulomb gas. Two vortices in the same layer are always separated by a distance of at least $\lambda_\parallel$, which may then be viewed as the effective "size" of a vortex. For a system with an infinite number of layers, each having a small but nonzero *density* of vortices, an infinitely long cylinder of radius $\lambda_\parallel$ perpendicular to and piercing the layers will "catch" an infinite number of vortices. The stack of pancakes caught by the cylinder may be viewed as an extended object. By densely packing the system with such cylinders, the configurations of the system may be viewed as configurations of these extended objects. The extended objects can be labelled by a species index $\mathbf{n} = (\ldots, n_1, n_2, \ldots)$ where n_i is an integer indicating the strength and sign of the vortex occupying layer i of the object in question (for a dilute system, most of the entries in $\mathbf{n}$ will be zero). We can formulate the problem in terms of these extended objects in which case the partition function Eq. (3.1) becomes:

$$\mathcal{Z} = \sum_{\{\mathbf{n}\}} \prod_{\mathbf{n}} y_{\mathbf{n}}^{N_{\mathbf{n}}} \sum_{\{c\}} \exp(-\beta \sum_{i\neq j} V_{\mathbf{n}_i,\mathbf{n}_j}),$$
(3.3)

where $\sum_{\{\mathbf{n}\}}$ is a sum over sets of extended objects consistent with charge neutrality in each layer and $\sum_{\{c\}}$ is over distinct spatial configurations of these objects. The interaction between two of these objects, indexed by $\mathbf{n}_i$ and $\mathbf{n}_j$, is logarithmic by construction and given by the sum of the pairwise interactions between the pancakes comprising each stack:

$$V_{\mathbf{n}_i,\mathbf{n}_j} = \sum_{k,l} V_{n_{ik},n_{jl}}$$
(3.4)

where k, l are layer indices and $V_{n_{ik},n_{jl}}$ is given by Eq. (2.3). Since an extended object is composed of an infinite number of pancakes, it will require an infinite creation energy $E_{\mathbf{n}}$ and the corresponding fugacity $y_{\mathbf{n}} = \exp(-\beta E_{\mathbf{n}})$ will be formally zero.[41] However, we can write the energy of an extended object as:

$$E_{\mathbf{n}} = \sum_{k} E_{n_k,k} + \sum_{k\neq l} E_{n_k n_l;kl}$$
(3.5)

where the first term is the creation energy of each pancake in the stack while the second is the pairwise interaction energy between different pancakes in the same stack. Eqs. (3.4) and (3.5) show that Eqs. (3.2) and (3.3) are equivalent ways of expressing the partition function.

Our goal is to determine the phase diagram of the model described by Eq. (3.2) (or Eq. (3.3)) in the dilute limit where each plane contains a small, but nonzero, density of vortices. The advantage of separating the vortex–vortex interaction into a long–distance logarithmic term, accounting for the short–distance physics through generalized fugacity variables, is that it permits the use of renormalization group (RG) techniques developed for studying the two–dimensional Coulomb gas.[26,28,30] The procedure requires us to consider a generalized version of Eq. (3.2):

$$\mathcal{Z} = \sum_{\{N_{k,l};N_{ab;ij}\ldots\}} \left(\prod_{k,l} y_{k,l}^{N_{k,l}} \prod_{ab;ij} w_{ab;ij}^{N_{ab;ij}} \prod_{abc;ijk} w_{abc;ijk}^{N_{abc;ijk}} \right.$$
$$\left. \prod_{abcd;ijkl} w_{abcd;ijkl}^{N_{abcd;ijkl}} \cdots \right) \sum_{\{c\}} \exp(-\beta \sum_{i\neq j}^{>} V_{ij}), \quad (3.6)$$

In this equation, $w_{abc;ijk} = \exp(-\beta E_{abc;ijk})$ where $E_{abc;ijk}$ is the three–body interaction energy of an aligned triplet of vortices where the first member is a vortex of type a in layer i, directly below the second member, a vortex of type b in layer j, which is directly below the third member, a vortex of type c in layer k. For equivalent layers, $E_{abc;ijk}$ will depend only on $|i-j|$ and $|j-k|$ (and the strengths a, b, and c). Similarly, we include fugacity variables corresponding to four (and higher) body interactions. These terms may be visualized in the extended object picture where Eq. (3.5) generalizes to:

$$E_{\mathbf{n}} = \sum_{k} E_{n_k,k} + \sum_{k\neq l} E_{n_k n_l;kl} + \sum_{k<l<m} E_{n_k n_l n_m;klm} + \cdots$$
(3.7)

In Appendices B and C, we present a detailed renormalization group treatment of this model. The analysis may be viewed as an iterative coarse–graining procedure connecting our model with a series of other models with the same critical properties. A physical picture of this procedure is illustrated and discussed in Fig. 1.

The analysis of Appendix C yields an infinite set of coupled flow equations for the extended object fugacities $\{y_{\mathbf{n}}\}$ and the analogous equations for the $(\{y_{k,l}\}, \{w_{ij;kl}\}, \ldots)$ variables. The system has a fixed point when the $\{y_{\mathbf{n}}\}$ variables are identically zero. A linearized theory about this fixed point (Eq. (C21)) suggests that at sufficiently low temperatures the fixed point is stable while at higher temperatures these variables become RG relevant. Beginning in the low temperature phase and raising the temperature, the phase transition is indicated by one of the $\{y_{\mathbf{n}}\}$ becoming marginal. Which fugacity is the first to "unbind" depends on the initial conditions of the RG flow.

If the starting model is given by Eq. (2.4), then the first extended objects to become marginal are those where $\mathbf{n}$ has +1 in one of its entries and zeroes everywhere else. By Eq. (3.7),
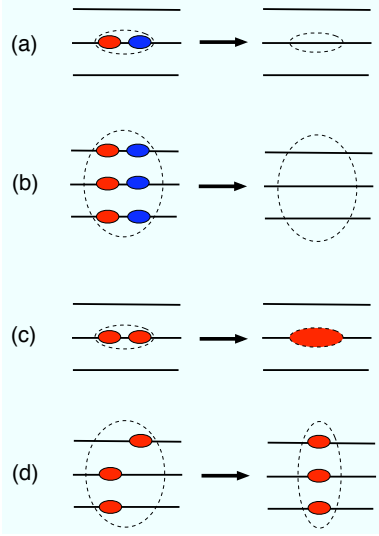
FIG. 1: Pictorial summary of the renormalization group procedure discussed in Appendices B and C. If an object and its anti-object are very closely spaced, then they will cancel upon coarse graining, but in the process the coupling constants will be renormalized. (a) shows the simplest such process involving single pancakes which may be viewed as integrating out the smallest loops of magnetic flux. (b) is an example of the more complicated object/anti–object cancellations which correspond to integrating out larger loops. Nearby objects that are not antiparticles will fuse together. (c) shows two single strength pancakes fusing into a double strength pancake. (d) shows a height two stack and a single pancake fusing into a height three stack.

the fugacity of this object is precisely the fugacity of a single strength pancake y ($\equiv y_{1,i}$ but since we have assumed a translationally invariant system, the layer index is not needed). The flow equation for this quantity, in the approximation where we keep only the leading y dependence, is:

$$\frac{dy^2}{d\epsilon} = y^2(4 - \beta q^2 \alpha_0). \tag{3.8}$$

In the same limit, the flow equations for the coupling constants are:

$$\frac{d(\beta q^2 \alpha_n)}{d\epsilon} = -\pi y^2 \sum_m (\beta q^2 \alpha_{n-m})(\beta q^2 \alpha_m). \tag{3.9}$$

where we have also taken the "distance to marginality", $(4 - \beta q^2 \alpha_0)$, as a small parameter. We will discuss the flow equations for the other variables further below.

Eqs. (3.8) and (3.9) are precisely the flow equations obtained in Ref. 24, using a formulation that involved only single pancakes, without accounting for extended objects. Our analysis implies that the more complicated objects are irrelevant at temperatures below the transition so in this limit, Eqs. (3.8) and (3.9) may have greater validity than initially suspected. As discussed in the Appendix C, this irrelevance does not simply follow from the linearized theory but involves using the sum rule (Eq. (2.5)) to place a bound on higher order terms – a technical issue that does not arise in the single–layer problem.

We continue our analysis by recognizing that the right side of Eq. (3.9) is a convolution of the couplings. Taking the Fourier transform, we obtain:

$$\frac{d(1/(\beta q^2 \alpha(k)))}{d\epsilon} = \pi y^2 \tag{3.10}$$

where $\alpha(k) = \sum_n \alpha_n e^{-ikn}$ and we used the fact that $\alpha_m = \alpha_{-m}$. Because the right side is independent of k, we may formally integrate this equation to obtain:

$$\beta q^2 \alpha(k, \epsilon) = \frac{\beta q^2 \alpha(k, 0)}{1 + \beta q^2 \alpha(k, 0)C(\epsilon)} \tag{3.11}$$

where $C(\epsilon) \equiv \int_0^\epsilon \pi y^2$ is an integration constant that obeys the flow equation:

$$\frac{dC}{d\epsilon} = \pi y^2 \tag{3.12}$$

with the initial condition $C(0) = 0$. Observe that $\alpha(0, \epsilon) = \sum_n \alpha_n$ so Eq. (3.11) implies that the sum rule is preserved by the flow.

Using Eqs. (3.8) and (3.12), we obtain a differential equation for the flow trajectories in the (C, y)-plane, which can be integrated:

$$y^2 = y_0^2 + \frac{1}{\pi}\left(4C - \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln(1 + \beta q^2 |\alpha(k, 0)C(l)|)\right) \tag{3.13}$$

Representative trajectories are sketched in Fig. 2 for a given value of y_0 as a function of temperature. As C increases from 0 monotonically during the flow (Eq. (3.12)), the system moves along these curves from left to right. At high temperatures, the fugacity increases monotonically during the flow. As the temperature is lowered, the fugacity initially decreases before increasing. At a critical temperature, the curve will intersect the $y = 0$ axis at one point and at lower temperatures, the curves cross the axis. However, $y = 0$ is a fixed point of Eqs. (3.8) and (3.9), which means the system will "stop" once $y = 0$ is reached. These curves demonstrate the existence of a low temperature phase, where the fugacity renormalizes to zero, separated from a high temperature region by a transition corresponding to the unbinding of $2d$ pancake vortices: this, by definition, is a Kosterlitz-Thouless transition.[28]

The most distinguishing characteristic of the single–layer KT transition is the universal jump in the superfluid stiffness.[31] To see what happens in the layered case, we need to determine the range of initial conditions for which the trajectory defined by Eq. (3.13) passes through $y = 0$ at some $C > 0$. That is, we need to solve:

$$\pi y_0^2 = \int \frac{dk}{2\pi} \ln(1 + \beta q^2 |\alpha(k, 0)|C) - 4C \tag{3.14}$$

If we begin close to the critical point, and assume the position of this point will only be changed by a small amount relative to the single–layer case, we can expand this expression treating

y_0, C, and $(\beta q^2 \alpha_0 - 4)$ as small parameters. Then, to leading order, we obtain:

$$\pi y_0^2 \approx (\beta q^2 \alpha_0(0) - 4)C - \frac{1}{4} \sum_m \alpha_m^2(0)C^2 \qquad (3.15)$$

where we have inverted the Fourier transform. In order to have a $y = 0$ solution where $C > 0$, the following criterion must be satisfied:

$$\beta q^2 \alpha_0(0) - 4 > \sqrt{\pi y_0^2 \sum_m \alpha_m^2(0)} \qquad (3.16)$$

Therefore, at temperatures $T > q^2\alpha_0(0)/4$, there is no solution and y will diverge as the flow coordinate $\epsilon \to \infty$. Eq. (3.11) indicates that the couplings $\{\alpha_n\}$ will go to zero in this same limit. The critical temperature at which a $y = 0$ fixed point exists is $T = q^2\alpha_0(0)/4$. This solution represents a critical surface ($y = 0$, $\{\alpha_n = \alpha_n(0)\}$) where the only constraints on the couplings $\{\alpha_n(0)\}$ are the sum rule (or, more precisely, that the matrix $\alpha_{ij} \equiv \alpha_{|i-j|}$ is positive definite — see the discussion in Appendix B) and that the values are such that the single strength pancake is the first fluctuation to become marginal.

To relate this to the universal jump, we need to relate the quantity $\alpha_0(0)$, given in Eq. (2.4), to the in–plane superfluid stiffness measured in an experiment. The in–plane superfluid stiffness is defined in terms of the in–plane magnetic penetration depth $\lambda_\parallel$, which is a directly measurable quantity: $\rho_s = \phi_0^2 s/(16\pi^3\lambda_\parallel^2) = q^2/2\pi$ (see the discussion between Eqs. (A1) and (A2)). As the critical temperature is crossed, the quantity $\beta q^2\alpha_0(\infty)$ jumps downward from 4 to zero. In terms of ρ_s:

$$\left[\frac{\rho_s\alpha_0(\infty)}{T}\right]_{T_c^-}^{T_c^+} = \frac{2}{\pi} \qquad (3.17)$$

$$\frac{[\rho_s]_{T_c^-}^{T_c^+}}{T_c} = \frac{2}{\pi\left(1 - \frac{s}{2\lambda_\parallel}\right)} \qquad (3.18)$$

Therefore, while the jump described by Eq. (3.17) is a universal quantity, the jump in the superfluid stiffness (Eq. (3.18)), which is the quantity that is directly measured, receives a non–universal correction on the order of 1 percent for $Bi_2Sr_2CaCu_2O_{8+x}$.

IV. LOW TEMPERATURE PHASE

Having established the existence of a critical point and phase transition, we now turn to the nature of the low temperature phase and the way the present case differs from a stack of decoupled layers. The most direct difference follows from Eq. (3.11). As the low temperature phase is characterized by a finite value of $C(\infty)$, Eq. (3.11) indicates that the interlayer
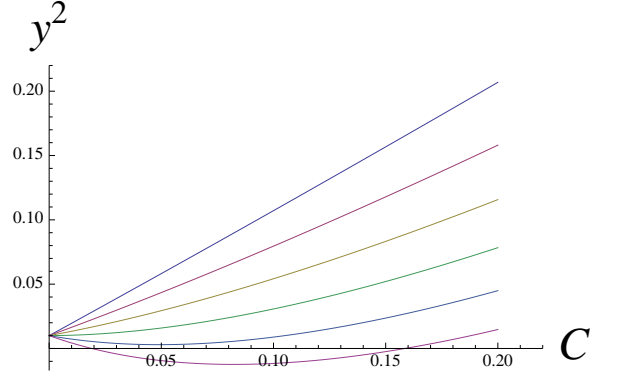

FIG. 2: Plot of the flow trajectories of Eq. (3.13) for the case where the initial fugacity $y_0 = 0.1$. The six curves, from top to bottom, are for $\beta q^2 = 1, 2, 3, 4, 5$, and 6 respectively. Since the flow stops when $y = 0$, this initial condition implies a phase transition occurring for βq^2 between 5 and 6.

couplings $\{\alpha_n\}$ will have nonzero fixed point values. Therefore, interlayer correlations will be present.

Another perspective may be gained by considering the flow equations for the pair interaction fugacity for single strength pancakes in layers 1 and m, which we label w_{1m} (Eq. (C25)):

$$\frac{dw_{1m}}{d\epsilon} = -\left[2 + \beta q^2\alpha_{m-1}\right]w_{1m} - \beta q^2\alpha_{m-1} \qquad (4.1)$$

where we have dropped terms that flow to zero in the low temperature phase as $\epsilon \to \infty$. Eq. (4.1) has a fixed point when:

$$w_{1m}(\infty) = \frac{\beta q^2|\alpha_{m-1}(\infty)|}{2 + \beta q^2|\alpha_{m-1}(\infty)|} \approx \frac{\beta q^2|\alpha_{m-1}(\infty)|}{2} \qquad (4.2)$$

In the decoupled limit, $w_{1m}(\infty)$ would be zero. The $y = 0$ fixed point means that at the longest length scales, the effective model is equivalent to one where the vortices are not present: i. e. the properties of the superconductor will be governed by the spin wave part of the action (Eq. (A11)). However, at intermediate length scales, which correspond to the points forming the RG trajectory, the effective model will be one where the vortices are present, albeit with small fugacity. w_{1m} flowing to zero means that configurations where vortices lie on top of one another contribute progressively less to the partition function, relative to configurations where they do not, as the system is viewed from larger length scales. The reason is purely entropic.[42] Eq. (4.2) indicates an enhanced probability for such configurations in the coupled layer case.

At one level, this is not so surprising because at the smallest length scales, the interlayer couplings (Eq. (2.4)) favor the formation of stacks. However, it is interesting to discuss the higher–body terms. For example, we may obtain the three-body interaction fugacity for single strength pancakes in layers a, b, and c, which we label w_{abc}. To obtain this quantity, it is easiest to start with the flow equation for $y_\mathbf{n} \equiv y_{abc}$ where $\mathbf{n}$ is a vector with $+1$ in layers a, b, and c, and 0 elsewhere.

From Appendix C, it follows, to leading order in y, that:

$$\frac{dy_{abc}}{d\epsilon} = \left[2 - \frac{\beta q^2}{2}\left(3\alpha_0 + \alpha_{ab} + \alpha_{ac} + \alpha_{bc}\right)\right]y_{abc}$$
$$- y\Big[y_{ab}(\alpha_{ac} + \alpha_{bc}) + y_{ac}(\alpha_{ab} + \alpha_{bc}) + y_{bc}(\alpha_{ab} + \alpha_{ac})\Big]$$
$$(4.3)$$

where y_{ij} is the fugacity of the extended object with single strength pancakes in layers i and j, the other layers being empty, and $\alpha_{ij} \equiv \alpha_{|i-j|}$. We can determine the low temperature fixed point value of y_{abc} by setting the right side of Eq. (4.3) to zero. By Eq. (3.7), these fugacities may be understood as a product of terms associated with the creation and interaction of the stacked vortices: $y_{ab} = y^2 w_{ab}$ and $y_{abc} = y^3 w_{ab} w_{ac} w_{bc} w_{abc} \equiv y^3 v_{abc}$. Using Eq. (4.2), we find:

$$v_{abc}(\infty)$$
$$= \frac{\frac{\beta q^2}{2}\left[(\alpha_{ab} + \alpha_{ac} + \alpha_{bc})^2 - (\alpha_{ab}^2 + \alpha_{ac}^2 + \alpha_{bc}^2)\right]}{\left(\frac{3\beta q^2}{2}\alpha_0 - 2\right) + \frac{\beta q^2}{2}\left(\alpha_{ab} + \alpha_{ac} + \alpha_{bc}\right)}\Bigg|_\infty$$
$$\approx \frac{\frac{\beta q^2}{2}\left[(\alpha_{ab} + \alpha_{ac} + \alpha_{bc})^2 - (\alpha_{ab}^2 + \alpha_{ac}^2 + \alpha_{bc}^2)\right]}{\left(\frac{3\beta q^2}{2}\alpha_0 - 2\right)}\Bigg|_\infty$$
$$(4.4)$$

In terms of the variable w_{abc}, the result is:

$$w_{abc}(\infty)$$
$$\approx \frac{\left(\frac{2}{\beta q^2}\right)^2 \left[(\alpha_{ab} + \alpha_{ac} + \alpha_{bc})^2 - (\alpha_{ab}^2 + \alpha_{ac}^2 + \alpha_{bc}^2)\right]}{\alpha_{ab}\alpha_{ac}\alpha_{bc}\left(\frac{3\beta q^2}{2}\alpha_0 - 2\right)}\Bigg|_\infty$$
$$(4.5)$$

In decoupled limit, v_{abc} and w_{abc} flow to zero and infinity respectively, which again is purely entropic. v_{abc} being nonzero indicates an enhanced probability for these objects in the coupled layer case. However, these expressions indicate that the effective interaction between the vortices within the stack is no longer a simple pairwise form. Qualitatively, this is another indication of three–dimensional correlations being present in the low temperature phase.

V. HIGH TEMPERATURES

Fig. 2 shows that the high temperature phase has $C \to \infty$ as $\epsilon \to \infty$. From Eq. (3.11), we conclude that the new fixed point model is one where all of the couplings have renormalized to zero. In the same limit, y also diverges, which may be interpreted as an unbinding of pancake vortices. Therefore, the high temperature phase may be viewed as a stack of nearly independent planes, each containing a neutral plasma of weakly interacting vortices. However, once y becomes of order unity, the renormalization group approach discussed in the previous Section no longer applies. The flow equations

(3.8) and (3.9) were obtained by dropping terms involving higher powers of y, which is no longer valid when y is order unity. Physically, the RG approach of Appendix C assumes a dilute gas of vortices, which is no longer true at high temperatures.

Therefore, to gain insight into the nature of the high temperature phase, a different analytical approach is needed. We study this limit using a Debye–Huckel mean field analysis where we assume each layer has N positive and N negative vortices, which we represent via density functions $\rho_m^+(\mathbf{x})$ and $\rho_m^-(\mathbf{x})$, where m is a layer index and $\mathbf{x}$ is the in–plane coordinate. The total charge density in layer m is given by $\rho_m(\mathbf{x}) = \rho_m^+(\mathbf{x}) - \rho_m^-(\mathbf{x})$. The system is modeled by the mean–field free energy functional:

$$\mathcal{F} = \frac{1}{2}\sum_{mn}\left[\int\int d^2x d^2y \rho_m(\mathbf{x})\rho_n(\mathbf{y})V_{mn}(\mathbf{x}-\mathbf{y})\right] +$$
$$T\sum_m\left[\int d^2x\left(\rho_m^+(\mathbf{x})\ln(\frac{\rho_m^+(\mathbf{x})}{N})) + \rho_m^-(\mathbf{x})\ln(\frac{\rho_m^-(\mathbf{x})}{N})\right)\right]$$
$$+ \sum_m\int d^2x \phi_m^{\text{ext}}(\mathbf{x})\rho_m(\mathbf{x})$$
$$(5.1)$$

where the three terms are the vortex–vortex interaction, system entropy, and interaction of the vortices with an external potential $\phi_m^{\text{ext}}(\mathbf{x})$. The potential $V_{mn}(\mathbf{x}-\mathbf{y})$ is given by Eq. (2.3). The idea is to minimize $\mathbf{F}$ with respect to the functions $\rho_m^\pm(\mathbf{x})$ subject to the constraint: $\int d^2x\rho_m^\pm(\mathbf{x}) = N$.

The previous paragraph is a natural way to discuss the high temperature limit of a layered Coulomb gas interacting via Eq. (2.3).[32] However, there is a caveat to note when we relate this model to superconductors. The phase transition discussed in the previous Section describes a loss of superconductivity, which is the low temperature state, due to a loss of long–range order in the *phase* of the order parameter. Therefore, in principle, one may have a regime without superconductivity but where the *amplitude* of the superconducting order parameter is nonzero.[3] It is in such a regime that our model applies since it is reasonable to assume the basic degrees of freedom would still be vortices. A nonzero order parameter amplitude means the effective penetration depth is finite so we expect the inter-layer mechanism of Eq. (2.3), which is ultimately due to the screening currents, to still apply. On the other hand, once the order parameter amplitude is zero, we can no longer think of the basic degrees of freedom as vortices so our model would no longer be relevant. In Ref. [3], it was suggested that for layered superconductors with small superconducting carrier densities, including the high–T_c materials, the energy scale associated with phase decoherence might be appreciably less than the energy scale at which the amplitude goes to zero. In such a case, there would be a temperature range above T_c where our model would apply.

In the absence of an external potential, the minimum free energy is obtained when both the positive and negative charges are uniformly distributed in each layer, i.e. $\rho_m^\pm(\mathbf{x}) = N/A \equiv \rho_0$ where A is the area of a layer. If we perturb the system about this limit with a small ϕ^{ext}, the corresponding

density fluctuation may be calculated as a linear response:

$$\delta\rho_m(\mathbf{x}) = -\sum_n \int d^2 x' \chi_{mn}(\mathbf{x} - \mathbf{x}')\phi_n^{\text{ext}}(\mathbf{x}') \qquad (5.2)$$

where the (Fourier transform) of the susceptibility $\chi_{mn}(\mathbf{x}, \mathbf{x}') \equiv -(\delta\rho_m(\mathbf{x})/\delta\phi_n^{\text{ext}}(\mathbf{x}'))_{\phi=0}$ is given by:

$$\chi(\mathbf{q}, k) = \frac{1}{\frac{T}{2\rho_0} + V(\mathbf{q}, k)} \qquad (5.3)$$

where $V(\mathbf{q}, k) = \alpha(k)\frac{2\pi}{q^2}$ is the Fourier transform of Eq. (2.3) and $\alpha(k)$ the Fourier transform of Eq. (2.4):

$$\begin{aligned}
\alpha(k) &= \sum_n \alpha(n) e^{-ikn} = \alpha_0 + \sum_{n\neq 0}\alpha_n e^{-ikn} \\
&= \frac{s}{2\lambda_\parallel}\left[\frac{e^{s/\lambda_\parallel}+1}{e^{s/\lambda_\parallel}-1}\right]\left[\frac{\cos k - 1}{\cos k - \cosh\frac{s}{\lambda_\parallel}}\right] \quad (5.4)
\end{aligned}$$

We choose $\phi^{\text{ext}}(\mathbf{x})$ to be the potential of a unit test charge at the origin of the $n = 0$ layer: $\phi^{\text{ext}}(\mathbf{q}, k) = \frac{2\pi}{q^2}\alpha(k)$ While this potential is not small near the origin, our interest is in the long distance behavior. The corresponding density fluctuation is:

$$\delta\rho(\mathbf{q}, k) = -\chi(\mathbf{q}, k)\phi^{\text{ext}}(\mathbf{q}, k) = \frac{\eta^2}{q^2 + \eta^2}\left[\frac{\cos k - 1}{\cos k - a_q}\right]$$

$$(5.5)$$

where $\eta^2 = \frac{4\pi\rho_0}{T}\frac{s}{2\lambda_\parallel}\left[\frac{e^{s/\lambda_\parallel}+1}{e^{s/\lambda_\parallel}-1}\right]$ and $a_q = \frac{q^2\cosh\frac{s}{\lambda_\parallel}+\eta^2}{q^2+\eta^2}$. Taking the inverse Fourier transform in the layering direction:

$$\delta\rho_m(\mathbf{q}) = \frac{\eta^2}{q^2 + \eta^2}\left[\delta_{m,0} - \sqrt{\frac{a_q-1}{a_q+1}}\left(a_q - \sqrt{a_q^2 - 1}\right)^m\right]$$

$$(5.6)$$

The inverse transform in the $\mathbf{q}$ direction is not a simple expression. However, if we expand in the small parameter $\frac{s}{\lambda_\parallel}$, and also assume that η is not too large, then to leading order, we obtain:

$$\begin{aligned}
\delta\rho_m(\mathbf{x}) &\approx \eta^2 \frac{e^{-\eta|\mathbf{x}|}}{\sqrt{\eta|\mathbf{x}|}}\left(\delta_{m,0} - \frac{s}{2\lambda_\parallel}e^{-\frac{|m|s}{\lambda_\parallel}}\right) \\
&= \eta^2 \frac{e^{-\eta|\mathbf{x}|}}{\sqrt{\eta|\mathbf{x}|}}\alpha_m \quad (5.7)
\end{aligned}$$

At large distances, the in- (out-of) plane fluctuation is positive (negative) which is expected by neutrality since a positive charge at the origin will attract negative (positive) charges towards the origin in the same (different) layers. The in–plane density fluctuation has essentially the same Yukawa form as the single layer problem. The screening length is slightly renormalized from the in–plane value of $\kappa = \sqrt{\frac{4\pi\rho_0}{T}}$ to

$\eta \approx \kappa(1 - \frac{s}{4\lambda})$. However, a more striking difference is that if we form an infinite stack by placing a test vortex at the origin of *every* layer, then the sum rule indicates that the stack will be completely screened while in the case of decoupled layers, a each test charge will influence a density fluctuation in its own layer of the 2d Yukawa form. In this sense, the high temperature phase of the infinite layer model does not correspond to a complete layer decoupling but retains some of its three–dimensional features.

VI. CONCLUSION

In conclusion, we have shown that the superconductor-to-normal phase transition in an infinite–layered, type–II superconductor, in the absence of Josephson coupling but in the presence of electromagnetic coupling, is a Kosterlitz–Thouless transition. The jump in the in–plane superfluid stiffness, which is a universal quantity in the single layer problem, acquires a small non–universal correction. We find that the phase transition is driven by the unbinding of two–dimensional pancake vortices but both the low and high temperature phases show three–dimensional characteristics.

A natural topic for future work is to find more connections with experiment, including ways to distinguish the electromagnetically coupled problem from the single layer case. A more pressing issue would be to explore how these conclusions are affected by having a small but nonzero Josephson coupling and/or other mechanisms of coupling the layers.

VII. ACKNOWLEDGEMENTS

APPENDIX A: RELATION OF GINZBURG-LANDAU AND COULOMB GAS MODELS

In this Section, we review how the Coulomb gas description of a layered superconductor arises from a phenomenological free energy functional of the Ginzburg-Landau type. Variants of this derivation may be found in a number of references, including the original paper of Efetov[2]. Our presentation closely follows Ref. 18.

Our starting point is the Lawrence-Doniach model of layered superconductors[33] where the system is modeled as a discrete set of superconducting layers stacked in the z direction. The layers are assumed to have the same thickness d and are uniformly spaced with interlayer separation s. With each layer n, we associate a superconducting order parameter $\Psi_n = |\Psi_n|e^{i\theta_n}$ which we assume does not vary in the z

direction within a layer. The Lawrence-Doniach free energy functional is then given by:

$$
\begin{aligned}
\mathcal{F}(\Psi_n, \mathbf{A}) \;=\; & d \sum_n \int d^3r\, \delta(z - z_n) \Big[\alpha |\Psi_n|^2 + \frac{\beta}{2} |\Psi_n|^4 + \frac{1}{2m_\parallel^*} \Big| \Big(-i\hbar\nabla_\parallel - \frac{e^*}{c} \mathbf{A}_\parallel \Big) \Psi_n \Big|^2 \\
& + \frac{\hbar^2}{2m_z^* s^2} \Big| \Psi_{n+1} \exp\Big(-i\frac{e^*}{c} \int_{ns}^{(n+1)s} dz\, A_z \Big) - \Psi_n \Big|^2 \Big] + \frac{1}{8\pi} \int d^3r (\nabla \times \mathbf{A})^2,
\end{aligned} \tag{A1}
$$

where the subscripts $\parallel$ and z refer to the two in-plane and one out-of-plane coordinates respectively and $e^* = 2e$ and $m_{\parallel,z}^* = 2m_{\parallel,z}$ are respectively the charge and effective masses of a Cooper pair. The difference between Eq. (A1) and the usual Ginzburg-Landau functional is that the order parameter fields are only defined within the layers so that the kinetic energy associated with the z direction is discretized. Note, however, that the magnetic field is defined everywhere in space. This approach differs from anisotropic Ginzburg-Landau theory, where the order parameter is defined everywhere and the z direction still has a continuum description. We expect Eq. (A1) to accurately describe highly anisotropic superconductors, such as the high-T_c compounds, but we are not aware of a precise way in which to derive Eq. (A1) as the limit of an anisotropic Ginzburg-Landau model.

Next, we assume the amplitude of the order parameter is constant in each layer, i.e. $|\Psi_n|^2 = n_s^* = \frac{n_s}{2}$, where n_s is the number of superconducting electrons per unit volume in a layer. This assumption clearly breaks down within the core of

a vortex, which is a region with a radius of order $\xi_\parallel \equiv \frac{\hbar^2}{2m_\parallel^* |\alpha|}$, the in-plane coherence length. For the type II superconductors of interest in the present work, $\xi_\parallel$ is small compared to the in-plane magnetic penetration depth $\lambda_\parallel \equiv (\frac{m_\parallel c^2}{4\pi \langle n_s \rangle e^2})^{1/2}$, which is the other in-plane length scale of interest; here $\langle n_s \rangle = n_s \frac{d}{s}$ is the average number density of superconducting electrons over the whole sample volume.[1] Therefore, if the smallest length scale we are interested in is of order $\lambda_\parallel$ and if we further assume that the concentration of vortices is dilute, it seems reasonable to neglect amplitude fluctuations.

The term in Eq. (A1) involving m_z may then be viewed as a Josephson coupling between the phase variables in adjacent layers. For the highly anisotropic materials which motivate the present work, $m_z \gg m_\parallel$ so we expect the Josephson coupling to be very small. In this paper, we assume the Josephson coupling is identically zero and hence ignore this term.

With these simplifications, Eq. (A1) leads to the following effective action:

$$
\mathcal{F}(\theta_n, \mathbf{A}) \;=\; \frac{\rho_s}{2} \int d^3r \sum_n \delta(z - ns)(\nabla_\parallel \theta_n - \frac{2\pi}{\phi_0} \mathbf{A}_\parallel)^2 + \frac{1}{8\pi} \int d^3r (\nabla \times \mathbf{A})^2 \tag{A2}
$$

where θ_n is the phase of the order parameter in layer n; $\rho_s = \frac{\hbar^2 n_s^* d}{m_\parallel^*}$ is the 2D superfluid stiffness of a layer; and $\phi_0 = \frac{hc}{2e}$. The interaction between layers is implicit in the second term.

The next step is to determine the $\mathbf{A}$ which minimizes the functional (A2). Once this is obtained, we can rewrite Eq. (A2) solely in terms of the order parameter. Taking the functional derivatives and imposing Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) gives the following equations:

$$
\nabla^2 A_z \;=\; 0 \tag{A3}
$$

$$
\nabla^2 \mathbf{A}_\parallel \;=\; \frac{1}{\Lambda} \sum_n \delta(z - ns)(\mathbf{A}_\parallel - \frac{\phi_0}{2\pi} \nabla_\parallel \theta_n) \tag{A4}
$$

where $\frac{1}{\Lambda} = 4\pi \rho_s (\frac{2\pi}{\phi_0})^2 = \frac{s}{\lambda_\parallel^2}$. Taking the Fourier transform

of Eq. (A4) gives:

$$
\mathbf{A}_\parallel(\mathbf{q}, k) = -\frac{\alpha_\parallel(\mathbf{q}, k) - \varphi_\parallel(\mathbf{q}, k)}{\Lambda(q^2 + k^2)} \tag{A5}
$$

where $\mathbf{q}$ and k are the momenta conjugate to the in-plane ($\mathbf{x}_\parallel$) and out-of-plane (z) coordinates and:

$$
\alpha_\parallel(\mathbf{q}, k) \;\equiv\; \sum_n e^{-ikns} \mathbf{A}_\parallel(\mathbf{q}, ns) = \frac{1}{s} \sum_m \mathbf{A}_\parallel(\mathbf{q}, k + \frac{2\pi m}{s}) \tag{A6}
$$

$$
\varphi_\parallel(\mathbf{q}, k) \;\equiv\; \sum_n e^{-ikns} \frac{\phi_0}{2\pi} \nabla_\parallel \theta_n(\mathbf{q}) \tag{A7}
$$

where $\mathbf{A}_\parallel(\mathbf{q}, ns)$ means that the Fourier transform is only in the in-plane direction. We can write the analog of Eq. (A5) with k replaced by $k + (2\pi m)/s$, where m is an integer.

Notice that $\alpha_\parallel(\mathbf{q}, k + (2\pi m)/s) = \alpha_\parallel(\mathbf{q}, k)$ and $\varphi_\parallel(\mathbf{q}, k + (2\pi m)/s) = \varphi_\parallel(\mathbf{q}, k)$. Therefore, summing both sides over m, and using Eq. (A6), we obtain the minimizing $\mathbf{A}$:

$$\mathbf{A}_\parallel(\mathbf{q}, k) = \frac{\varphi_\parallel(\mathbf{q}, k)}{(q^2 + k^2)(\Lambda + L(\mathbf{q}, k))}$$

$$(A8)$$

$$A_z(\mathbf{q}, k) = -\frac{\mathbf{q} \cdot \varphi_\parallel(\mathbf{q}, k)}{k(q^2 + k^2)(\Lambda + L(\mathbf{q}, k))}$$

$$(A9)$$

where the last expression follows from the gauge constraint

and

$$L(\mathbf{q}, k) \equiv \frac{1}{s} \sum_m \frac{1}{q^2 + (k + \frac{2\pi m}{s})^2}$$

$$= \frac{1}{2q} \frac{\sinh qs}{\cos ks - \cosh qs} \qquad (A10)$$

where the last identity is obtained by (standard) complex analysis methods. Substituting Eqs. (A8) and (A9) into Eq. (A2) will give an effective action in terms of the order parameter itself. After some tedious but straightforward algebra, one obtains:

$$\mathcal{F} = \frac{\phi_0^2}{32\pi^3\Lambda} \sum_{m,n} \int \frac{d^2q}{(2\pi)^2} \left[\left(\frac{\delta_{m,n}|\mathbf{q} \times \nabla_\parallel \theta_n(\mathbf{q})|^2}{q^2} - \int \frac{dk}{2\pi} \frac{[\mathbf{q} \times \nabla_\parallel \theta_m(\mathbf{q})][\mathbf{q} \times \nabla_\parallel \theta_n(\mathbf{q})]e^{ik(m-n)s}}{q^2(q^2 + k^2)(\Lambda + L(\mathbf{q}, k))} \right) \right.$$
$$\left. + \left(\frac{\delta_{m,n}|\mathbf{q} \cdot \nabla_\parallel \theta_n(\mathbf{q})|^2}{q^2} - \int \frac{dk}{2\pi} \frac{[\mathbf{q} \cdot \nabla_\parallel \theta_m(\mathbf{q})][\mathbf{q} \cdot \nabla_\parallel \theta_n(\mathbf{q})]e^{ik(m-n)s}}{q^2(q^2 + k^2)(\Lambda + L(\mathbf{q}, k))} \left(1 - \frac{q^2\Lambda}{k^2(\Lambda + L(\mathbf{q}, k))} \right) \right) \right] \qquad (A11)$$

The two sets of large "()" brackets indicate that the free energy functional decouples into two parts which depend respectively on the divergence-free and irrotational parts of the fields $\{\nabla_\parallel \theta_n\}$. These correspond respectively to the "vortex" and "spin-wave" excitations of the order parameter field, which appear in the conventional analysis of the $2d$XY model.[28] We make the usual assumption that the phase diagram is determined by the vortex part so will ignore the spin waves from now on.

We make the identification $2\pi\rho_v(\mathbf{q}, n) \equiv \mathbf{q} \times \nabla_\parallel \theta_n(\mathbf{q})$ where $\rho_v(\mathbf{x}, n)$ is the vortex number density per unit area in layer n.[43] After performing the k integration[44], the vortex part of the free energy is given by:

$$\mathcal{F}_v = \frac{1}{2} \sum_{m,n} \int d^2x d^2y \rho_v(\mathbf{x}, m)\rho_v(\mathbf{y}, n)V_{mn}(\mathbf{x} - \mathbf{y})$$

$$(A12)$$

where the vortex-vortex interaction is given by:

$$V_{mn}(\mathbf{x}) = \frac{\phi_0^2 s}{4\pi\lambda_\parallel^2} \int \frac{d^2q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{q^2}[\delta_{mn} - W_{mn}(\mathbf{q})] \quad (A13)$$

and

$$W_{mn}(\mathbf{q}) = \frac{s\sinh(qs)}{2\lambda_\parallel^2 q} \frac{(G_q - (G_q^2 - 1)^{1/2})^{|m-n|}}{(G_q^2 - 1)^{1/2}} \quad (A14)$$

where $G_q = \cosh(qs) + \frac{s\sinh(qs)}{2\lambda_\parallel^2 q}$. While this interaction has a complicated appearance, our interest is primarily in its long distance (relative to $\lambda_\parallel$) behavior, which was obtained in Ref. 22:

$$V_{mn}(|\mathbf{x}|) \simeq -\frac{\phi_0^2 s}{8\pi^2\lambda_\parallel^2}\alpha_{|m-n|}\ln|x| \quad (A15)$$

where:

$$\alpha_n \simeq \delta_{n,0} - \frac{s}{2\lambda_\parallel}e^{-\frac{s}{\lambda_\parallel}|n|} \quad (A16)$$

and the "$\simeq$" sign indicates that this result is to leading order in $\frac{s}{\lambda_\parallel}$, which is assumed to be a small parameter. For a discrete set of vortices, $\rho_v(\mathbf{x}, n) = \sum_i m_i\delta(\mathbf{x} - \mathbf{x_i})$ where $m_i = \pm 1, \pm 2, \ldots$ and $\mathbf{x_i}$ are the strength and position respectively of the ith vortex in layer n. If, in addition to the interaction energy in Eq. (A15), we also assume that each vortex has a self-energy associated with its core, which we may represent through a fugacity, we arrive at the partition function given by Eq. (3.1).

APPENDIX B: RELATION OF COULOMB GAS AND SINE GORDON MODELS

The generalized Coulomb gas model discussed in the text may be formulated as a sine Gordon field theory, as in the two–dimensional case.[34]:

$$S[\{z_\mathbf{n}\}, \phi] = -\sum_{i,j} \frac{g_{ij}}{2} \int d^2x \nabla\phi_i \cdot \nabla\phi_j$$
$$+ \sum_\mathbf{n} \frac{z_\mathbf{n}}{\tau^2} \int d^2x \cos[\mathbf{n} \cdot \phi(\mathbf{x})]. \quad (B1)$$

In this Appendix, we establish the equivalence of this expression with Eqs. (3.3) and (3.6). In Appendix C, we analyze the phase diagram of this action using the renormalization group. Here i, j are layer indices and τ is a short distance cutoff of order the in–plane penetration depth $\lambda_\parallel$. The matrix $\mathbf{g}$ is defined

so its inverse $\mathbf{g}^{-1}$ is related to the coupling matrix for the layered Coulomb gas, given in Eq. (2.4): $g_{ij}^{-1}/2\pi = \beta q^2 \alpha_{|i-j|}$. The factor $z_{\mathbf{n}} = 2y_{\mathbf{n}}$, where $y_{\mathbf{n}}$ is the fugacity of an extended object indexed by the occupation vector $\mathbf{n}$ as discussed in the text. The vector $\boldsymbol{\phi} = (\ldots, \phi_1, \phi_2, \ldots)$ where ϕ_i is a sine–Gordon field corresponding to layer i.

We begin by writing Eq. (B1) as a partition function $\mathcal{Z} = \int D\boldsymbol{\phi} e^{S[\{z_{\mathbf{n}}\},\boldsymbol{\phi}]}$ and expanding the cosine terms:

$$\mathcal{Z} = \mathcal{Z}_0 \Big\langle \prod_{\mathbf{n}} \Big[1 + \frac{z_{\mathbf{n}}}{\tau^2} \int d^2x \cos(\mathbf{n} \cdot \boldsymbol{\phi}(\mathbf{x}))$$
$$+ \frac{z_{\mathbf{n}}^2}{2!\tau^4} \int\int d^2x d^2y \cos(\mathbf{n}\cdot\boldsymbol{\phi}(\mathbf{x}))\cos(\mathbf{n}\cdot\boldsymbol{\phi}(\mathbf{y})) + \ldots \Big] \Big\rangle_0.$$
$$(B2)$$

Here $\mathcal{Z}_0 = \int D\boldsymbol{\phi} e^{S_0[\boldsymbol{\phi}]}$ where $S_0[\boldsymbol{\phi}] = -\sum_{i,j} \frac{g_{ij}}{2} \int \nabla\phi_i \cdot \nabla\phi_j$ and $\langle \ldots \rangle_0$ denotes an average over the *full* Gaussian measure. The Coulomb gas partition function follows from calculating these averages. We begin by writing the cosine terms as:

$$\cos[\mathbf{n} \cdot \boldsymbol{\phi}(\mathbf{x})] = \frac{1}{2} \sum_{\mathbf{n}(\mathbf{x})=\pm\mathbf{n}} e^{i\mathbf{n}(\mathbf{x})\cdot\boldsymbol{\phi}(\mathbf{x})}. \qquad (B3)$$

The averages involve calculating expressions like $\langle \exp[i\sum_{a=1}^N \mathbf{n}_a(\mathbf{x}_a) \cdot \boldsymbol{\phi}(\mathbf{x}_a)]\rangle_0$. Performing the Gaussian integral:

$$\langle e^{[i\sum_{a=1}^N \mathbf{n}_a(\mathbf{x}_a)\cdot\boldsymbol{\phi}(\mathbf{x}_a)]}\rangle_0$$
$$= \exp\Big(-\frac{1}{2}\sum_{kl} N_k N_l \langle\phi_k(0)\phi_l(0)\rangle$$
$$-\frac{1}{2}\sum_{abkl} n_{ak}(\mathbf{x}_a)n_{bl}(\mathbf{x}_b)(\langle\phi_k(\mathbf{x}_a)\phi_l(\mathbf{x}_b)\rangle - \langle\phi_k(0)\phi_l(0)\rangle)\Big)$$
$$(B4)$$

where k,l are layer indices; a,b are indices denoting the N objects $\{\mathbf{n}_a\}$ entering the average; $N_k = \sum_a n_{ka}$ is the total vorticity in layer k due to these N objects; and the two-point function is:

$$\langle\phi_k(x)\phi_l(y)\rangle = g_{kl}^{-1} \int \frac{d^2q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{q^2} \qquad (B5)$$

Notice that $\langle\phi_k(0)\phi_l(0)\rangle = \frac{g_{kl}^{-1}}{2\pi} \ln\frac{L}{\tau} + \ldots$. Hence the first sum in the exponential is:

$$-\frac{1}{2}\sum_{kl} N_k N_l \langle\phi_k(0)\phi_l(0)\rangle = -\frac{1}{2}\ln\frac{L}{\tau}\mathbf{N}^{\mathrm{T}}\mathbf{g}^{-1}\mathbf{N} + \ldots$$
$$(B6)$$

where "$\ldots$" are terms subleading in L. The form of the interaction (Eq. (2.4)) and the sum rule (Eq. (2.5)) ensure that the coupling matrix $\mathbf{g}^{-1}$ is positive definite.[45] Therefore, $\mathbf{N}^{\mathrm{T}}\mathbf{g}^{-1}\mathbf{N} > 0$, implying the average in Eq. (B4) will be zero as $L \to \infty$ *unless* $\mathbf{N} = \mathbf{0}$. Therefore, the only configurations of objects $(\mathbf{n}_a, \cdots, \mathbf{n}_N)$ with nonzero expectation value are those that satisfy vortex neutrality in *each* layer, a stronger

condition than overall vortex neutrality. If this constraint is met, then Eq. (B4) becomes:

$$\exp[-\frac{1}{2}\sum_{a,b,k,l} n_{ak}(\mathbf{x}_a)n_{bl}(\mathbf{x}_b)V_{kl}(\mathbf{x_a}-\mathbf{x_b})] \qquad (B7)$$

where k and l indicate the planes where pancakes with strengths n_{ak} and n_{bl} and in–plane coordinates $\mathbf{x}_a$ and $\mathbf{x}_b$ reside and:

$$V_{kl}(\mathbf{x_a}-\mathbf{x_b}) = \langle\phi_k(\mathbf{x_a})\phi_l(\mathbf{x_b})\rangle - \langle\phi_k(0)\phi_l(0)\rangle$$
$$= -\frac{g_{kl}^{-1}}{2\pi}\ln\frac{|\mathbf{x_a}-\mathbf{x_b}|}{\tau} + \ldots \qquad (B8)$$

where "..." are terms that are subleading for large $|\mathbf{x_i} - \mathbf{x_j}|$.

The term we are considering will also have a prefactor. Suppose the N objects entering the average in Eq. (B4) include $N_{\mathbf{n}_a}$ objects of type $\mathbf{n}_a$ for $a = 1, \ldots, N$. There will then be a factor of $\prod_{a=1}^N [z_{\mathbf{n}_a}^{N_{\mathbf{n}_a}}/(N_{\mathbf{n}_a})!]$ from the Taylor expansion and a factor of $1/2^N$ from writing the N cosines as exponentials. Eq. (B4) is the integrand of a $2N$ dimensional spatial integral, which may be viewed as summing over different spatial configurations of these N objects. If we choose to write this as a sum over *indistinguishable* configurations, then there will also be a factor $\prod_{a=1}^N (N_{\mathbf{n}_a})!$ for the number of configurations (contained in the integral) identical to having the objects $\{\mathbf{n}_a\}$ at spatial positions $\{\mathbf{x}_a\}$. Combining everything (and dropping the unimportant factor $\mathcal{Z}_0$): we may rewrite Eq. (B1):

$$\mathcal{Z} = \sum_{\{\mathbf{n}\}} \prod_{\mathbf{n}} (z_{\mathbf{n}}/2)^{N_{\mathbf{n}}} \sum_{\{c\}} \exp(-\sum_{i\neq j} V_{\mathbf{n}_i,\mathbf{n}_j}) \qquad (B9)$$

where the first sum is over sets of objects $\{\mathbf{n}\}$ satisfying vortex neutrality in each layer and the second is over indistinguishable spatial configurations $\{c\}$ of these objects. Comparing Eqs. (B7)–(B9) with Eq. (3.3), we see that the sine Gordon theory is equivalent to our generalized Coulomb gas if we make the following identifications:

$$y_{\mathbf{n}} = z_{\mathbf{n}}/2 \qquad (B10)$$

$$\beta q^2 \alpha_{|i-j|} = g_{ij}^{-1}/2\pi \qquad (B11)$$

as asserted. Using Eq. (3.7) to expand the fugacities, we see that the sine–Gordon model of Eq. (B1) is also identical to the partition function in Eq. (3.6).

The final point to note is that Eq. (B7) takes the logarithmic form at distances large compared to τ but will vanish as $|\mathbf{x}_a - \mathbf{x}_b| \to 0$. In contrast, the interaction that enters Eqs. (3.3) and (3.6) is a hard–core interaction (the particles are assumed to be at least distance τ laterally apart). As discussed in Ref. [26], this is not actually a problem as a sine–Gordon model literally equivalent to a hard–core Coulomb gas is possible with a slight renormalization of the coupling constants g_{ij} that will not affect our calculations (since these corrections will manifest as higher order terms in the RG analysis).

APPENDIX C: DERIVATION OF FLOW EQUATIONS

In this Section, we derive the RG flow equations presented in the main text. Our starting point is the action for the layered sine Gordon model (Eq. (B1)) which, as shown in Appendix B, is equivalent to the layered Coulomb gas (Eq. (3.3)) discussed in the text. We analyze this action using an extension of the momentum shell renormalizaton group approach discussed in Ref. 26. The calculation applies in the small fugacity limit.

The first step is to write the action (B1) as $S[\{z_\mathbf{n}\}, \phi] = S_0[\phi] + S_1[\{z_\mathbf{n}\}, \phi]$, where S_0 is the Gaussian term. The fields $\{\phi_i\}$ are written as a sum of fast and slow modes, i.e. $\phi_i = \phi_{i,<} + \phi_{i,>}$ where:

$$\phi_{i,<}(\mathbf{x}) = \int_{|\mathbf{q}|\in[0,\frac{\Lambda}{s}]} \frac{d^2q}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}}\phi_i(\mathbf{q}) \qquad (C1)$$

$$\phi_{i,>}(\mathbf{x}) = \int_{|\mathbf{q}|\in[\frac{\Lambda}{s},\Lambda]} \frac{d^2q}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}}\phi_i(\mathbf{q}) \qquad (C2)$$

Here $\phi_i(\mathbf{q})$ is the Fourier transform of $\phi_i(\mathbf{x})$, $\Lambda \sim \frac{1}{\tau}$ is an ultraviolet cutoff, and $s = 1 + \epsilon$ is a rescaling parameter. The idea is to integrate over the fast modes to get an effective action for $\phi_<(\mathbf{x})$. The Gaussian term separates to give: $\mathcal{Z} = \int \mathcal{D}\phi_< e^{S_0[\phi_<]} \int \mathcal{D}\phi_> e^{S_0[\phi_>]} e^{S_1[\{z_\mathbf{n}\},\phi_<,\phi_>]} = A \int \mathcal{D}\phi_< e^{S_0[\phi_<]} \langle e^{S_1[\{z_\mathbf{n}\},\phi_<,\phi_>]}\rangle_>$ where the constant $A = \int \mathcal{D}\phi_> e^{S_0[\phi_>]}$ will be dropped because it does not affect the critical properties of the model. The subscript $(>)$ on the average, which we will also drop, denotes that the Gaussian average is over the fast modes only.

We can write the average as $\langle e^{S_1[\{z_\mathbf{n}\},\phi]}\rangle = e^{S_1'[\{z_\mathbf{n}'\},\phi_<]}$ where the relation between S_1 and S_1' may be expressed as a cumulant expansion:

$$S_1' = \langle S_1\rangle + \frac{1}{2}(\langle S_1^2\rangle - \langle S_1\rangle^2) + \dots$$
$$= \sum_\mathbf{n} \frac{z_\mathbf{n}}{\tau^2}\int d^2x\langle\cos(\mathbf{n}\cdot\phi(\mathbf{x}))\rangle + \sum_{\mathbf{m},\mathbf{n}} \frac{z_\mathbf{m}z_\mathbf{n}}{2\tau^4}$$
$$\int\int d^2xd^2y\Big[\langle\cos(\mathbf{m}\cdot\phi(\mathbf{x}))\cos(\mathbf{n}\cdot\phi(\mathbf{y}))\rangle$$
$$- \langle\cos(\mathbf{m}\cdot\phi(\mathbf{x}))\rangle\langle\cos(\mathbf{n}\cdot\phi(\mathbf{y}))\rangle\Big]$$
$$+ \dots. \quad (C3)$$

A convenient way to obtain these averages is with the useful fact:

$$\langle e^{\int d^2x\mathbf{J}(\mathbf{x})\cdot\phi(\mathbf{x})}\rangle = e^{\int d^2x\mathbf{J}(\mathbf{x})\cdot\phi_<(\mathbf{x})}$$
$$\times \exp\Big[\sum_{k,l}\frac{1}{2}\int\int d^2xd^2yJ_k(\mathbf{x})\langle\phi_{k,>}(\mathbf{x})\phi_{l,>}(\mathbf{y})\rangle J_l(\mathbf{y})\Big]$$
$$(C4)$$

where the two–point function is:

$$\langle\phi_{k,>}(\mathbf{x})\phi_{l,>}(\mathbf{y})\rangle = g_{kl}^{-1}\int_{|\mathbf{q}|\in[\frac{\Lambda}{s},\Lambda]} \frac{d^2q}{(2\pi)^2}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{q^2} \quad (C5)$$

Using these relations with $\mathbf{J}(\mathbf{x}') = i\mathbf{n}\delta(\mathbf{x}' - \mathbf{x})$, we readily obtain:

$$\langle\cos[\mathbf{n}\cdot\phi(\mathbf{x})]\rangle = s^{-\frac{1}{4\pi}\sum_{kl} g_{kl}^{-1}n_kn_l}\cos[\mathbf{n}\cdot\phi_<(\mathbf{x})] \quad (C6)$$

and with $\mathbf{J}(\mathbf{x}') = i\mathbf{n}\delta(\mathbf{x}' - \mathbf{x}) \pm i\mathbf{m}\delta(\mathbf{x}' - \mathbf{y})$:

$$\langle\cos[\mathbf{m}\cdot\phi(\mathbf{x})]\cos[\mathbf{n}\cdot\phi(\mathbf{y})]\rangle - \langle\cos[\mathbf{m}\cdot\phi(\mathbf{x})]\rangle\langle\cos[\mathbf{n}\cdot\phi(\mathbf{y})]\rangle$$
$$= \frac{1}{2}s^{-\frac{1}{4\pi}\sum_{kl}g_{kl}^{-1}(m_km_l+n_kn_l)}\Big([e^{-\frac{1}{2}\sum_{kl}(m_kn_l+m_ln_k)\langle\phi_{k,>}(\mathbf{x})\phi_{l,>}(\mathbf{y})\rangle} - 1]\cos[\mathbf{m}\cdot\phi_<(\mathbf{x}) + \mathbf{n}\cdot\phi_<(\mathbf{y})]$$
$$+ [e^{\frac{1}{2}\sum_{kl}(m_kn_l+m_ln_k)\langle\phi_{k,>}(\mathbf{x})\phi_{l,>}(\mathbf{y})\rangle} - 1]\cos[\mathbf{m}\cdot\phi_<(\mathbf{x}) - \mathbf{n}\cdot\phi_<(\mathbf{y})]\Big)$$
$$\approx \frac{1}{2}\Big(\frac{1}{2}\sum_{kl}(m_kn_l + m_ln_k)\langle\phi_{k,>}(\mathbf{x})\phi_{l,>}(\mathbf{y})\rangle\Big)\Big(\cos[\mathbf{m}\cdot\phi_<(\mathbf{x}) - \mathbf{n}\cdot\phi_<(\mathbf{y})] - \cos[\mathbf{m}\cdot\phi_<(\mathbf{x}) + \mathbf{n}\cdot\phi_<(\mathbf{y})]\Big)$$
$$(C7)$$

where in the last line, we retained terms to linear order in ϵ.

As the ultraviolet cutoff $\Lambda \to \infty$, the integral (C5) becomes arbitrarily small except when $x \approx y$. Therefore, we make the approximation: $\langle\phi_{i,>}(\mathbf{x})\phi_{j,>}(\mathbf{y})\rangle \approx \tau^2\delta(\mathbf{x} - \mathbf{y})\langle\phi_{i,>}(0)\phi_{j,>}(0)\rangle \approx \tau^2\delta(\mathbf{x} - \mathbf{y})\frac{g_{ij}^{-1}}{2\pi}\epsilon$ and replace the cosine operators with the leading term in their operator product expansions.[35] For $\mathbf{m} \neq \mathbf{n}$, this implies the replacement:

$$\cos[\mathbf{m}\cdot\phi_<(\mathbf{x}) \pm \mathbf{n}\cdot\phi_<(\mathbf{y})] \approx \cos[(\mathbf{m}\pm\mathbf{n})\cdot\phi_<(\mathbf{x})].$$
$$(C8)$$

Physically, this means that two closely-spaced vortex stacks, when viewed from a distance, appear as a single stack com-

posed of pancakes that are fusions of those in the two stacks. When $\mathbf{m} = \mathbf{n}$, one of the terms in Eq. (C7) is of this fusion type:

$$\cos[\mathbf{m} \cdot (\boldsymbol{\phi}_<(\mathbf{x}) + \boldsymbol{\phi}_<(\mathbf{y}))] \approx \cos[2\mathbf{m} \cdot \boldsymbol{\phi}_<(\mathbf{x})], \quad \text{(C9)}$$

while the other involves a nontrivial operator identification:

$$\cos[\mathbf{m} \cdot (\boldsymbol{\phi}_<(\mathbf{x}) - \boldsymbol{\phi}_<(\mathbf{y}))] \approx -\frac{1}{4} \sum_{ij} m_i m_j \nabla \phi_i \cdot \nabla \phi_j. \quad \text{(C10)}$$

Physically, this latter term means that two identical, closely-spaced vortex stacks of opposite sign, to leading order, screen one another at long distances. However, the effect of this screening is a renormalization of the interlayer couplings.

Using these expressions, we may write an equation for $S_1'[\{z_\mathbf{n}'\}]$ to second order in the fugacities:

$$S_1'[\{z_\mathbf{n}'\}, \boldsymbol{\phi}_<] = -\sum_{i,j} \frac{\epsilon \gamma_{ij}}{2} \int d^2 x \nabla \phi_{i,<} \cdot \nabla \phi_{j,<}$$
$$+ \sum_\mathbf{n} \frac{z_\mathbf{n}'}{\tau^2} \int d^2 x \cos[\mathbf{n} \cdot \boldsymbol{\phi}_<(\mathbf{x})]. \quad \text{(C11)}$$

where:

$$\gamma_{ij} = \frac{1}{16\pi} \sum_\mathbf{m} z_\mathbf{m}^2 \left(\sum_{kl} m_k m_l g_{kl}^{-1} \right) m_i m_j \quad \text{(C12)}$$

and

$$z_\mathbf{n}' = \left[z_\mathbf{n} - \frac{\epsilon}{8\pi} \sum_\mathbf{m} z_{\mathbf{n}-\mathbf{m}} z_\mathbf{m} \left(\sum_{kl} (n_k - m_k) m_l g_{kl}^{-1} \right) \right.$$
$$\left. + \frac{\epsilon}{8\pi} \sum_\mathbf{m} z_{\mathbf{n}+\mathbf{m}} z_\mathbf{m} \left(\sum_{kl} (n_k + m_k) m_l g_{kl}^{-1} \right) \right]$$
$$\times \left(1 - \frac{\epsilon}{4\pi} \sum_{kl} n_k n_l g_{kl}^{-1} \right). \quad \text{(C13)}$$

The final step is to restore the ultraviolet cutoff of the original problem by rescaling the length. The net effect of doing this, to leading order in ϵ, is an additional multiplicative factor $(1 + 2\epsilon)$ on the right side of Eq. (C13). Combining Eq. (C11) with the Gaussian term S_0, we obtain an action similar to Eq. (B1) but with renormalized parameters. We can use Eqs. (C11)–(C13) to write flow equations for these parameters:

$$\frac{dz_\mathbf{n}}{d\epsilon} = \left(2 - \frac{1}{4\pi} \sum_{kl} n_k n_l g_{kl}^{-1} \right) z_\mathbf{n}$$
$$- \frac{1}{8\pi} \sum_\mathbf{m} z_{\mathbf{n}-\mathbf{m}} z_\mathbf{m} \left(\sum_{kl} (n_k - m_k) m_l g_{kl}^{-1} \right)$$
$$+ \frac{1}{8\pi} \sum_\mathbf{m} z_{\mathbf{n}+\mathbf{m}} z_\mathbf{m} \left(\sum_{kl} (n_k + m_k) m_l g_{kl}^{-1} \right). \quad \text{(C14)}$$

and:

$$\frac{dg_{ij}}{d\epsilon} = \frac{1}{16\pi} \sum_\mathbf{m} z_\mathbf{m}^2 \left(\sum_{kl} m_k m_l g_{kl}^{-1} \right) m_i m_j. \quad \text{(C15)}$$

It is more convenient to write the latter in terms of the inverse coupling matrix:[46]

$$\frac{dg_{ij}^{-1}}{d\epsilon} =$$
$$- \frac{1}{16\pi} \sum_\mathbf{m} z_\mathbf{m}^2 \left(\sum_{kl} m_k m_l g_{kl}^{-1} \right) \left(\sum_{pq} g_{ip}^{-1} m_p m_q g_{qj}^{-1} \right). \quad \text{(C16)}$$

Eqs. (C14)–(C16) are natural generalizations of the usual Coulomb gas flow equations.[30] However, as mentioned in the text, the fugacity $z_\mathbf{n}$ of an extended object $\mathbf{n}$ can be expressed in terms of the creation and interaction energies of the pancakes forming the stack. This latter formulation is convenient because in this picture, the fundamental degrees of freedom are individual pancakes, which have finite energy, as opposed to extended objects, which have (formally) infinite energy. The content of Eqs. (C14)–(C16) can be recast in this language via the following relation which follows from Eq. (3.7):

$$\frac{z_\mathbf{n}}{2\pi} = \prod_i y_{n_i;i} \prod_{i<j} w_{n_i,n_j;i,j} \prod_{i<j<k} w_{n_i,n_j,n_k;i,j,k} \cdots. \quad \text{(C17)}$$

where i, j, k are layer indices and the y and w variables were defined in the text. This implies:

$$\frac{dz_\mathbf{n}}{z_\mathbf{n}} =$$
$$\sum_i \frac{dy_{n_i;i}}{y_{n_i;i}} + \sum_{i<j} \frac{dw_{n_i,n_j;i,j}}{w_{n_i,n_j;i,j}} + \sum_{i<j<k} \frac{dw_{n_i,n_j,n_k;i,j,k}}{w_{n_i,n_j,n_k;i,j,k}}$$
$$+ \dots. \quad \text{(C18)}$$

The flow equation for the y variables are obtained by considering Eq. (C14) for the occupation vector $\mathbf{n}_{a;i}$ which has only one nonzero entry: a strength a vortex in layer i. For a translationally invariant system, the fugacity variable will not depend on our choice of layer i so $y_{a;i} = y_a$ and:

$$\frac{dy_a}{y_a} = \frac{dz_{\mathbf{n}_{a;i}}}{z_{\mathbf{n}_{a;i}}} = 2 - \frac{a^2}{4\pi} g_{00}^{-1}$$
$$+ \frac{1}{8\pi} \sum_\mathbf{m} \frac{z_{\mathbf{n}_{a;i}-\mathbf{m}} z_\mathbf{m}}{z_{\mathbf{n}_{a;i}}} \left(\sum_{kl} (a\delta_{k,0} - m_k) m_l g_{kl}^{-1} \right). \quad \text{(C19)}$$

Using Eq. (C19), we can obtain the flow equations for the two–body fugacities $w_{ab;ij}$ by considering Eq. (C14) with occupation vector $\mathbf{n}_{ab;ij}$ which has only two nonzero entries cor-

responding to vortices of strengths a and b in layers i and j.

$$\frac{dw_{ab;ij}}{w_{ab;ij}} = \frac{dz_{\mathbf{n}_{ab;ij}}}{z_{\mathbf{n}_{ab;ij}}} - \left(\frac{dy_a}{y_a} + \frac{dy_b}{y_b}\right) = -\left(2 + \frac{ab}{2\pi}g_{ij}^{-1}\right)$$

$$-\frac{1}{8\pi}\sum_{\mathbf{m}} z_{\mathbf{m}}\left[\left(\frac{z_{\mathbf{n}_{ab;ij}-\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}} - \frac{z_{\mathbf{n}_{ab;ij}+\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}} - \frac{z_{\mathbf{n}_{a;i}-\mathbf{m}}}{z_{\mathbf{n}_{a;i}}} + \frac{z_{\mathbf{n}_{a;i}+\mathbf{m}}}{z_{\mathbf{n}_{a;i}}}\right)\right.$$

$$\times \left(a\sum_l m_l g_{il}^{-1}\right) + \left(\frac{z_{\mathbf{n}_{ab;ij}-\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}} - \frac{z_{\mathbf{n}_{ab;ij}+\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}} - \frac{z_{\mathbf{n}_{b;j}-\mathbf{m}}}{z_{\mathbf{n}_{b;j}}}\right.$$

$$\left. + \frac{z_{\mathbf{n}_{b;j}+\mathbf{m}}}{z_{\mathbf{n}_{b;j}}}\right)\left(b\sum_l m_l g_{jl}^{-1}\right) - \left(\frac{z_{\mathbf{n}_{ab;ij}-\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}} - \frac{z_{\mathbf{n}_{ab;ij}+\mathbf{m}}}{z_{\mathbf{n}_{ab;ij}}}\right.$$

$$\left.\left. - \frac{z_{\mathbf{n}_{a;i}-\mathbf{m}}}{z_{\mathbf{n}_{a;i}}} + \frac{z_{\mathbf{n}_{a;i}+\mathbf{m}}}{z_{\mathbf{n}_{a;i}}} - \frac{z_{\mathbf{n}_{b;j}-\mathbf{m}}}{z_{\mathbf{n}_{b;j}}} + \frac{z_{\mathbf{n}_{b;j}+\mathbf{m}}}{z_{\mathbf{n}_{b;j}}}\right)\left(\sum_{kl} m_k m_l g_{kl}^{-1}\right)\right]. \quad \text{(C20)}$$

In this manner, we can systematically obtain the flow equations for each of the many–body fugacities, though the formal expressions become increasingly complicated.

From Eqs. (C14)–(C16), we see that the system has a fixed point when all of the $z_{\mathbf{n}}$'s equal 0. To probe the stability of this fixed point, we consider Eq. (C14), keeping only the linear term for the moment. Converting to the variables in the main text via Eqs. (B10)–(B11), we have:

$$\frac{dy_{\mathbf{n}}}{d\epsilon} = \left(2 - \frac{\beta}{2}\left[q^2 \sum_{kl} n_k n_l \alpha_{|k-l|}\right]\right) y_{\mathbf{n}}. \quad \text{(C21)}$$

where the term in square brackets is the coefficient of the logarithmic interaction energy between two stacks of pancake vortices characterized by the same occupation vector $\mathbf{n}$. From Eq. (2.4), it may be shown that this quantity is always nonnegative and strictly positive for occupation vectors $\mathbf{n}$ that are "compact", i.e. whose non-zero entries all occur in a region of finite extent. For compact vectors, Eq. (C21) indicates the corresponding fugacities will be irrelevant at zero temperature. As the temperature is raised, the first fugacities to become marginal correspond to the vectors $\{\mathbf{n}_{1;i}\}$, which have a unit strength pancake in one layer, the other layers being empty. Similar considerations apply for non-compact vectors that are sparsely filled. The magnitude of the square bracket term can, in principle, be lowered by considering non-compact vectors that are densely filled. For example, in the case where we have a unit strength pancake in *every* layer of an infinite system, the square bracket term will vanish by the sum rule. However, in a dilute system, such objects will not be present in the initial model.

Therefore, in the case which concerns us, the stability of the fixed point is determined by the $\{y_{\mathbf{n}_{1;i}}\}$, which via Eq. (C17) is equivalent to the single strength pancake fugacity $y_1 \equiv y$ governed by the equation:

$$\frac{dy}{d\epsilon} = (2 - \frac{\beta q^2}{2}\alpha_0)y + O(y^3) \quad \text{(C22)}$$

We can similarly approximate the other flow equations by keeping only the leading powers in y. Because we are interested in the critical behavior, we take the "distance to marginality" $x \equiv (\beta q^2 \alpha_0 - 4)$ as an additional small parameter. In such an expansion, Eq. (C16) becomes:

$$\frac{d(\beta q^2 \alpha_n)}{d\epsilon} = -\pi y^2 \sum_p (\beta q^2 \alpha_{n-p})(\beta q^2 \alpha_p) \quad \text{(C23)}$$

The simplest composite object is a pair of single strength vortices, the first in layer 1 and the second directly above in layer m. The flow equation for the corresponding fugacity $y_{\mathbf{n}_{11;1m}} \equiv y_{1m}$ simplifies to:

$$\frac{dy_{1m}}{d\epsilon} = \left[2 - \beta q^2\left(\alpha_0 + \alpha_{m-1}\right)\right] y_{1m} - \beta q^2 \alpha_{m-1} y^2 \quad \text{(C24)}$$

The only fusion term surviving the expansion is the simplest one: single strength pancakes in layers 1 and m combining to form the object y_{1m}. The equation for the two–body fugacity $w_{11;1m} \equiv w_{1m}$ becomes:

$$\frac{dw_{1m}}{d\epsilon} = -\left[2 + \beta q^2 \alpha_{m-1}\right] w_{1m} - \beta q^2 \alpha_{m-1} \quad \text{(C25)}$$

In this manner, we can obtain simplified versions of the full flow equations appropriate for the physical limit that interests us.

There is an important technical point implicit in this procedure. In Eq. (C22), we have asserted that the terms we have ignored are $O(y^3)$. From Eq. (C14), we can see that an example of such a term is $g_{11}^{-1} y_{1m} y$. This arises from the fusion of the extended object composed of two unit strength positive pancakes in layers 1 and m with a unit strength negative pancake in layer m. Since we expect $y_{1m} \sim y^2$, this term will be of order y^3. However, there are an *infinite* number of such terms which, in total, could potentially overwhelm the "leading term" whenever y is nonzero. This differs from the usual single plane Coulomb gas where there are only a finite number of $O(y^3)$ processes (such as, for example, the fusion of a $+2$ and -1 pancake as well as terms related to the cutoff procedure[30]). To see this is not a problem, note that in the low temperature phase, Eq. (C24) implies that $y_{1m} \sim |\alpha_m|y^2/2$. The infinite sum converges due to the sum rule: $\sum_{m\neq 1} y_{1m} y \sim y^3 \sum_{m\neq 1} |\alpha_m| = \alpha_0 y^3$. The argument is similar for higher order processes. In the absence of such a sum rule, the pile–up of an infinite number of higher order terms can overwhelm the leading term and hence invalidate the analysis.[47]

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[37] However, a jump in the superfluid stiffness characteristic of a KT transition has been seen in very thin films of underdoped $YBa_2Cu_3O_{6+x}$.[36]

[38] More precisely, since the rotational symmetry of the plane is broken, we would expect the pancakes to be anisotropic in shape, since the coherence length will be different whether one is parallel or perpendicular to the stripes. Also, the interactions themselves might be anisotropic. However, we expect the leading long–distance behavior to be at least qualitatively captured by an isotropic model.

[39] A common way of writing Eq. (3.1) is with the sum over $\{c\}$ replaced by a multidimensional integral over the coordinates of the vortices[24,28] in which case factors $1/(N_{k,l})!$ are needed because the vortices are indistinguishable.

[40] More precisely, the core energy $E_{k,l}$ will no longer represent a "bare" core energy but a renormalized energy that includes the effects of screening due to closely spaced vortex–antivortex pairs. This screening will also renormalize the coefficients $\{\alpha_n\}$ of the logarithmic long–distance interaction from the values in Eq. (2.4). For a sufficiently dilute system of vortices, we expect these renormalizations to be very small. Therefore, we will still use the values in Eq. (2.4) as initial conditions for the RG analysis of Eq. (3.2).

[41] There is no inconsistency because any physical quantity will involve an average where a substantial part of Eq. (3.3) will appear in both the numerator and denominator and hence cancel.

[42] A coarse–graining step may be viewed as taking the effective core size of a vortex from τ to $\tau + d\tau$. A vortex in the new model represents $(1 + \frac{d\tau}{\tau})^2$ configurations of the old model. This is accounted for by renormalizing the fugacity. Now consider an extended object y_{1m}, with pancakes in layers 1 and m. In the coarse–grained model, this object will still represent $(1 + \frac{d\tau}{\tau})^2$ configurations of the old model but merely increasing the size of the pancakes will give an extra factor of $(1 + \frac{d\tau}{\tau})^2$. Hence, because $y_{1m} = y^2 w_{1m}$, we can account for this overcounting by renormalizing w_{1m} by the factor $(1 - \frac{d\tau}{\tau})^2$.

[43] To see this, note that a unit strength vortex at the origin is represented by the phase field $\nabla\theta(\mathbf{x}) = \frac{\hat{\phi}}{|\mathbf{x}|}$. For a contour C encircling the singularity, $\oint_C \nabla\theta \cdot \vec{dl} = 2\pi$. In the continuum case, this becomes $\oint_C \nabla\theta \cdot \vec{dl} = 2\pi \int \rho_v(\mathbf{x})dA$. However, by Green's theorem, we know that $\oint_C \nabla\theta \cdot \vec{dl} = \int (\nabla \times \nabla\theta) \cdot \vec{dA}$. The identification $2\pi\rho_v(\mathbf{x}) = (\nabla \times \nabla\theta) \cdot \hat{z}$ follows.

[44] To do the integral, it is helpful to first break the range $[-\infty, \infty]$ into separate integrals over the ranges $[-3\pi/s, \pi/s]$, $[-\pi/s, \pi/s]$, $[\pi/s, 3\pi/s]$ and so on. In the integral over $[(2a - 1)\pi/s, (2a + 1)\pi/s]$, make the variable change $k \to k + 2\pi a/s$, so that all integrals are from $[-\pi/s, \pi/s]$. Using Eq. (A10) and the fact that $L(\mathbf{q}, k + 2\pi a/s) = L(\mathbf{q}, k)$, the separate integrals can be combined into one which can be performed by contour integration over the unit circle.

[45] To see this, note that one test of whether a symmetric matrix is positive definite is if (a) its diagonal entries are positive and (b) the diagonal entry is larger than the sum of the absolute values of the other entries in its row (or column). A second test is that a symmetric matrix is positive definite if and only if all of the leading principal minors of the matrix are positive (i.e. which means

the following matrices have positive determinants: the upper left 1-by-1 column, the upper left 2-by-2 column, and so on up to the matrix itself). The sum rule, and the fact that the magnitudes of the couplings monotonically decrease with interlayer distance, ensures, by the first test, that for a finite number of layers, the matrix $\mathbf{g}^{-1}$ is positive definite. As these are the leading minors of the infinite matrix, the second test ensures that the positive–definiteness holds in the infinite layer case as well.

[46] To obtain this equation, it is helpful to start from the infinitesimal expression for $\mathbf{g}$: $\mathbf{g}' = \mathbf{g} + \epsilon\boldsymbol{\gamma} = \mathbf{g}(1 + \epsilon\mathbf{g}^{-1}\boldsymbol{\gamma})$, where the matrix $\boldsymbol{\gamma}$ is given by Eq. (C12). Then, to leading order in ϵ: $\mathbf{g}^{-1} = (1 - \epsilon\mathbf{g}^{-1}\boldsymbol{\gamma})\mathbf{g}^{-1} = \mathbf{g}^{-1} - \epsilon\mathbf{g}^{-1}\boldsymbol{\gamma}\mathbf{g}^{-1}$.

[47] This pile–up issue is sometimes relevant in theories of sliding Luttinger liquids. E. Fradkin, private communication.